# ON THE STABILITY OF A TRIVIAL SOLUTION OF A LINEAR SYSTEM WITH PERIODIC COEFFICIENTS

## (OB USTOICHIVOSTI TRIVIAL'NOGO RESHENIIA LINEINYKH SISTEM S PERIODICHESKIMI KOEFFITSIENTAMY)

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We shall write down a system of $n$ linear differential equations of the first order with periodic coefficients in the form of one vector differential equation

$$\frac{dx}{dt} = A(t)\, x \tag{0.1}$$

Here $x$ is a vector function, $A(t)$ is a periodic (with a period $\omega > 0$) matrix function with real sectionally continuous elements:

$$x = \begin{pmatrix} x \\ \vdots \\ x_n \end{pmatrix}, \qquad A(t) = \| a_{ij}(t) \|_1^n, \qquad a_{ij}(t+\omega) = a_{ij}(t) \qquad (i, j = 1, \ldots, n)$$

It is well known [1,2] that the stability of a trivial solution of the system (0.1) depends on the roots of the corresponding characteristic equation

$$\det [X(\omega) - \rho I_n] = (-1)^n \{\rho^n - a_1\rho^{n-1} + \ldots + (-1)^{n-1} a_{n-1}\rho + (-1)^n \alpha\} = 0 \tag{0.2}$$

where $X(t)$ is the fundamental matrix of the system (0.1), $X(0) = I_n$ ($I_n$ is a unit matrix of $n$-th order). The trivial solution of (0.1) is stable if all roots of the equation (0.2) are less than or equal to unity in magnitude, or, in the case of multiple roots which eqaul unity in magnitude, if there exist simple elementary divisors of the matrix $X(\omega) - \rho I_n$. In the opposite case, that is, if at least one of the roots is greater than unity in magnitude, or when a multiple root in absolute value equalling unity is accompanied by a non-simple elementary divisor of the above mentioned matrix, then the trivial solution of the system (0.1) is unstable.

The constant term in equation (0.2) is given by the Liouville formula

$$\alpha = \det X(\omega) = \exp \int_0^\omega \operatorname{sp} A(t)\, dt \qquad \left(\operatorname{sp} A(t) = \sum_{i=1}^n a_{ii}(t)\right)$$

If the following inequality ([2], pp. 8 and 46)

$$\int_0^\omega \operatorname{sp} A(t)\, dt > 0$$

is satisfied, then det $X(t) \to +\infty$ when $t \to +\infty$, and the trivial solution of the system (0.1) is unstable. Therefore, we shall assume in what follows that

$$\int_0^\omega \operatorname{sp} A(t)\, dt \leqslant 0$$

It is well known ([3], p. 63) that if all eigenvalues of the symmetric matrix

$$A(t) + A^\tau(t) = \| a_{ij}(t) + a_{ji}(t) \|_1^n$$

for any $t (0 \leqslant t \leqslant \omega)$ are non-positive (or positive), then the trivial solution of the system (0.1) is stable, (or unstable).

Let us now consider how to determine the regions of stability and instability in the spatial coefficients of equation (0.2).

1. Let us begin with the system (0.1) written in the canonical form [4]

$$\frac{dx}{dt} = J_{2m} H(t)\, x, \qquad J_{2m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}, \qquad H(t) = \| h_{ij}(t) \|_1^{2m}$$

$$h_{ij} = h_{ji} \qquad (i, j = 1, \ldots, 2m) \tag{1.1}$$

If the characteristic equation for a system in canonical form happens also to be reciprocal, then we could expand it as follows

$$\rho^{2m} - a_1 \rho^{2m-1} + a_2 \rho^{2m-2} - \ldots + (-1)^m a_m \rho^m + (-1)^{m-1} a_{m-1} \rho^{m-1} + \ldots$$
$$\ldots + a_2 \rho^2 - a_1 \rho + 1 = 0 \tag{1.2}$$

When $m = 1$ then the region of stability is the interval $-2 < a_1 < 2$ and the regions of instability are the intervals $-\infty < a_1 < -2$ and $2 < a_1 < \infty$, whereas the points $a_1 = +2$ require some additional investigation [2].

When $m = 2$, that is, for a canonical system of the fourth order, the region of stability is defined by the inequalities (see Liapunov [5] p.8)

$$-2 < a_2 < 6,$$
$$4\,(a_2 - 2) < a_1^2 < \tfrac{1}{4}(a_2 + 2)^2 \tag{1.3}$$

and is shaded in Fig. 1.

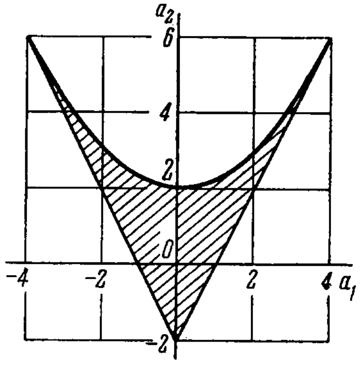


Fig. 1.

The region of instability is the region outside the curvilinear triangle, and the values of coefficients $a_1$ and $a_2$ which correspond to the edges of the triangle require additional investigation. It should be mentioned that the region $a_1 < -4$, $a_1 > 4$, $a_2 < -2$, $a_2 > 6$ is in the region of instability.

When $m \geqslant 3$ the problem is much more complicated. Reference [6] gives a survey of the necessary procedures: here we shall present only a conclusion of the Herglotz theorem [7].

According to Herglotz, the necessary and sufficient conditions for non-repeated roots equal to unity in magnitude of the real symmetric polynomial (1.2) are obtained from the positive quadratic form

$$\sum_{k,\,l=0}^{2m-1} s_{k-l}\xi_k\xi_l \qquad (s_{l-k} = s_{k-l})$$

where $s_k$ is the Newton sum.* To obtain results (1.3) from this quadratic form, for example, requires more work than is shown in [5].

The method proposed by Liapunov [5], and also the method of conformal mapping of the inside of a unit circle in a complex plane $\rho$ into the left-half plane, both show that in order to have all roots of the equation (1.2) equal to unity in magnitude the roots of certain $n$-th degree equa-

* Translator's note: Newton sum is $s_k = \sum_i^{2m} \rho_i^{\,k}$, where $\rho_i$ is a root of (1.2).

tion must be real, and the coefficients of this equation would be linear combinations of the coefficients of the equation (1.2). When $m = 3$, the above condition will define a certain part of a complicated third-order surface in the space of coefficients $a_1 a_2 a_3$.

An analog of the square shown in Fig. 1, is a hyperparallelepiped in the space $a_1 a_2, \ldots, a_m$ with edges parallel to the coordinate hyperplanes which could be determined relatively easily. Let us proceed to do so.

Let us first determine, in the equation (1.2), the intervals of variation of the coefficient $a_\mu (\mu = 1, \ldots, m)$ such that if $a_\mu$ is in the interval, then the trivial solution of the system (1.1) would be unstable, whatever the values of the remaining $m - 1$ coefficients. Since the coefficients $a_\mu$ is the sum

$$a_\mu = \sum \rho_{j_1} \cdots \rho_{j_\mu}$$

where $\rho_j (j = 1, 2, \ldots, 2m)$ are roots of the equation (1.2), and the sum is taken over all combinations of indices of the sequence 1, 2, ..., $2m$, it is clear that the interval

$$C_{2m}^{\mu} < a_\mu < \infty \tag{1.4}$$

($C_{2m}^{\mu}$ is the number of combinations of $2m$ elements taken $\mu$ at a time) is one of the required intervals.

If $\mu$ is odd

$$\mu = 2\nu - 1 \qquad (\nu = 1, \ldots, \text{ent } \tfrac{1}{2}(m + 1))$$

then the interval

$$-\infty < a_{2\nu-1} < -C_{2m}^{2\nu-1} \tag{1.5}$$

is another one of the required intervals.

The lower limit in (1.4) is the greatest lower bound [infimum], (the upper limit in (1.5) is the least upper bound [supremum]); hence if $\rho_1 = \ldots = \rho_{2m} = 1$ ($\rho_1 = \ldots = \rho_{2m} = -1$) and the elementary divisors of the matrix $X(\omega) - \rho I_{2m}$ are simple, then the trivial solution of the system (1.1) is stable and

$$a_\mu = C_{2m}^{\mu} \quad (a_{2\nu-1} = -C_{2m}^{2\nu-1})$$

Determination of intervals similar to (1.5) for even coefficients is not as simple as in the case of the odd ones.

Let $\mu = 2$, and let us find the smallest value of the coefficient

$$a_2 = \rho_1\rho_2 + \cdots + \rho_1\rho_{2m} + \rho_2\rho_3 + \cdots + \rho_2\rho_{2m} + \cdots + \rho_{2m-1}\rho_{2m}$$

such that all the roots of the equation (1.2) ($m \geqslant 2$) equal unity in magnitude. Let $2k$ roots equal $(-1)$ and $2m-2k$ roots equal unity. The number of negative roots must be even because the constant term in the equation (1.2) equals unity. Without loss of generality we can assume that $2k \leqslant m$, because if $2k > m$, then we could change signs of all the roots, which would leave the sign of $a_2$ unchanged. Thus, the number of terms in the expression for $a_2$ which equals unity is $C_{2k}^2 \quad C_{2m-2k}^2$ and $4k(m-k)$ terms equal $(-1)$, and

$$a_2 = C_{2k}{}^2 + C_{2m-2k}^2 - 4k(m-k) = 8k^2 - 8mk + 2m^2 - m$$
$$\frac{da_2}{dk} = 8(2k-m)$$

The smallest value of $a_2$ corresponds to $2k = m$, when $m$ is even, or to $2k = m-1$, when $m$ is odd. When $2k = m$, we have $a_2 = -m$, and when $2k = m-1$, we have $a_2 = -(m-2)$. Thus, if the following inequality

$$a_2 < -m + 1 - (-1)^m \tag{1.6}$$

is satisfied, the trivial solution of the system (1.1) is unstable irrespective of the values of the remaining coefficients in the equation (1.2).

Let us apply the inequalities (1.4), (1.5) and (1.6) to the system (1.1) of the sixth order, whose corresponding equation (1.2) has the form

$$\rho^6 - a_1\rho^5 + a_2\rho^4 - a_3\rho^3 + a_2\rho^2 - a_1\rho + 1 = 0$$

Let us construct the parallelepiped:

$$-6 \leqslant a_1 \leqslant 6, \qquad -1 \leqslant a_2 \leqslant 15, \qquad -20 \leqslant a_3 \leqslant 20$$

The region outside the parallelepiped belongs to the region of instability; that is, if any of the six inequalities

$$a_1 < -6, \qquad a_1 > 6, \qquad a_2 < -1, \quad a_2 > 15, \qquad a_3 < -20, \qquad a_3 > 20$$

is satisfied, the trivial solution of the system (1.1) (when $m = 3$) is unstable. None of the six inequalities considered separately could be sharpened, because the region of stability touches the constructed parallelepiped at each of its six edges.

The attempt to obtain inequalities similar to (1.6) for the remaining even coefficients was unsuccessful and numerical methods had to be used.

Let $\mu = 4$ and let us find the smallest value of the coefficient

$$\overline{a_4} = \rho_1\rho_2\rho_3\rho_4 + \ldots + \rho_{2m-3}\rho_{2m-2}\rho_{2m-1}\rho_{2m}$$

such that all roots of the equation (1.2) $(m \geqslant 4)$ equal unity in magnitude. Let $2k$ roots $(2k \leqslant m)$ equal $(-1)$, and $2m-2k$ roots equal $(+1)$. Let the number of terms equalling $(-1)$ in the expression for $a_4$ be $N_{2k}$. Then

$$a_4 = -N_{2k} + (C_{2m}^4 - N_{2k}) = C_{2m}^4 - 2N_{2k}$$

and the problem consists of finding the greatest value of $N_{2k}$ over all natural numbers $k$ satisfying the inequality $2k \leqslant m$. It is readily seen that

$$N_2 = C_2^1 C_{2m-2}^{3}, \qquad N_4 = C_4^1 C_{2m-4}^{3} + C_4^3 C_{2m-4}^{1}, \ldots$$

$$N_{2k} = C_{2k}^1 C_{2m-2k}^{3} + C_{2k}^3 C_{2m-2k}^{1} = \tfrac{4}{3}(-k^2 + mk)(4k^2 - 4mk + 2m^2 - 3m + 2)$$

The greatest value of $N_{2k}$ for $m$ = 4, 5, 6, 7, 8, 9 and also the smallest value of $a_4$ as obtained from computations are tabulated below:

| | | | | | | |
|---|---|---|---|---|---|---|
| $m =$ | 4 | 5 | 6 | 7 | 8 | 9 |
| $(N_{2k})$ gr $=$ | 40 | 112 | 256 | 520 | 928 | 1560 |
| $2k =$ | 2 | 2 | 4 | 4 | 4 | 6 |
| $(a_4)$ sm $=$ | −10 | −14 | −17 | −36 | −36 | −60 |

In the general case for an even coefficient $a_{2\nu}$ the problem consists in finding the greatest value of $N_{2k}$ over all natural numbers $k$ satisfying the inequality $2k \leqslant m$. The expressions for $N_{2k}$ are

$$N_2 = C_2^1 C_{2m-2}^{2\nu-1}, \quad N_4 = C_4^1 C_{2m-4}^{2\nu-1} + C_4^3 C_{2m-4}^{2\nu-3}$$

$$N_6 = C_6^1 C_{2m-6}^{2\nu-1} + C_6^3 C_{2m-6}^{2\nu-3} + C_6^5 C_{2m-6}^{2\nu-5}$$

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$$N_{2k} = C_{2k}^1 C_{2m-2k}^{2\nu-1} + C_{2k}^3 C_{2m-2k}^{2\nu-3} + \ldots + C_{2k}^{2k-1} C_{2m-2k}^{2\nu-2k+1}$$

2. The equation (0.2) will now be investigated in greater generality, assuming that $0 < a < 1$, and regarding $a = 1$ as the limiting case. The conditions

$$|a_\nu| \leqslant C_n^\nu \qquad (\nu = 1, \ldots, n-1) \tag{2.1}$$

are necessary for the stability of the solution of the system (0.1), because if any one of these conditions is not satisfied, at least one of the roots of the equation (0.2) will be greater than unity in magnitude. It should be mentioned that even more rigid conditions than (2.1) could be obtained.

Let us limit our investigation to finding the interval of variation of the coefficient $a_1$, for which the trivial solution of the system (0.1) is unstable irrespective of the values of $a_2, a_3, \ldots, a_{n-1}$. The product of the roots $\rho_1, \rho_2, \ldots, \rho_n = a$; therefore, we can assume

$$\rho_1 = \ldots = \rho_{n-k} = 1,$$

$$\rho_{n-k+1} = \ldots = \rho_n = \sqrt[k]{\alpha} \qquad (k = 1, \ldots, n)$$

which gives the value of $a_1$ as

$$a_1 = \rho_1 + \ldots + \rho_n = n - k + k\sqrt[k]{\alpha} \tag{2.2}$$

We shall prove the following inequality

$$n - k + k\sqrt[k]{\alpha} > n - (k+1) + (k+1)\sqrt[k+1]{\alpha} \qquad (k = 1, \ldots, n-1)$$

The above inequality can be written in the form

$$f(z) \equiv kz^{k+1} - (k+1)z^k + 1 > 0 \qquad (z = \alpha^{\frac{1}{k(k+1)}},\ 0 < z < 1)$$

which is obviously valid, because

$$f(0) = 1, \qquad f(1) = 0, \qquad f'(z) = -k(k+1)z^{k-1}(1-z) < 0 \qquad (0 < z < 1)$$

Thus, under the conditions

$$|\rho_\nu| \leqslant 1 \quad (\nu = 1, \ldots, n), \qquad \rho_1 \cdots \rho_n = \alpha \tag{2.3}$$

the greatest value of $a_1$ is given* by the expression (2.2) when $k = 1$, and it equals $n - 1 + a$.

The least value of $a_1$ under conditions (2.3) for $n$ even equals $1 - a - n$, and for $n$ odd equals $1 + a - n$. Thus, if

$$a_1 < -n + 1 - (-1)^n\alpha \quad \text{or} \quad a_1 > n - (1 - \alpha) \tag{2.4}$$

then the trivial solution of the system (0.1) is unstable.

The necessary and sufficient conditions for the roots of equation (0.2) to lie inside unit circle are givenby Schur's theorem [8,9] (which does not apply to equation (1.2) ). It should be mentioned that Schur's algorithm is almost as difficult as solving the Hurwitz problem.

As an illustration, let us show the regions of stability in the spatial coefficients of equation (0.2) when $n = 2$ and $n = 3$.

When $n = 2$, the region of stability is defined by the inequalities $0 < a \leqslant 1$, $|a_1| \leqslant 1 + a$ (the points $a = 1$, $a_1 = +2$ must be investigated separately). When $n = 3$, the region of stability is defined by the

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* Probably, under conditions (2.3), the values $\rho_1 = \ldots = \rho_{n-1} = 1$, $\rho_n = a$ define also the greatest values of the remaining coefficients in equation (0.2).

inequalities $a_2 > a_1 - 1 + \alpha$, $a_2 < \alpha a_1 + 1 - \alpha^2$, $a_2 > -a_1 - \alpha - 1$, and for a fixed $\alpha$ it is represented by the interior of a triangle in the plane $a_1$, $a_2$ (Fig. 2). The perimeter of the triangle must be investigated separately.

3. Following Liapunov [2,10], let us, together with the system (0.1), investigate an auxiliary system

$$dx/dt = \varepsilon A(t) x \tag{3.1}$$

and seek the fundamental matrix $X(t, \epsilon)$ in the form of a series

$$X(t, \varepsilon) = X_0(t) + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + \ldots, \qquad X_0(0) = I_n$$
$$X_k(0) = 0 \qquad (k = 1, 2, \ldots) \tag{3.2}$$

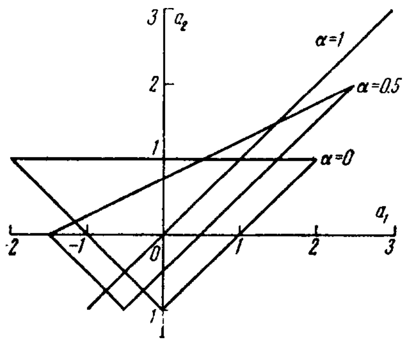


Fig. 2.

Substituting (3.2) in (3.1) and equating terms of the same power in $\epsilon$, we obtain a sequence of matrix differential equations:

$$\frac{dX_0}{dt} = 0, \qquad \frac{dX_k}{dt} = A(t) X_{k-1} \qquad (k = 1, 2, \ldots)$$

which can be integrated for given initial condition, yielding

$$X_0(t) = I_n, \qquad X_k(t) = \int_0^t A(t_1) X_{k-1}(t_1) dt_1 \qquad (k = 1, 2, \ldots)$$

Liapunov proved that the matrix series (3.2) is absolutely convergent for all values of $\epsilon$ and is also uniformly convergent in every finite interval of variation of $t$. By substituting $\epsilon = 1$, we could express the fundamental matrix of the system (0.1) as:

$$X(\omega) = X(\omega;\ 1) = I_n + X_1(\omega) + X_2(\omega) + \ldots$$

where

$$X_1(\omega) = \int_0^{\omega} A(t_1)\,dt_1$$

$$X_k(\omega) = \int_0^{\omega} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{k-1}} A(t_1)\ldots A(t_k)\,dt_k \qquad (k = 2, 3, \ldots) \quad (3.3)$$

The first coefficient of equation (0.2) equals

$$a_1 = \mathrm{sp}\, X(\omega) = n + a_1^{(1)} + \ldots + a_1^{(k)} + \ldots \quad (3.4)$$

where

$$a_1^{(1)} = \mathrm{sp}\int_0^{\omega} A(t_1)\,dt_1, \qquad a_1^{(k)} = \mathrm{sp}\int_0^{\omega} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{k-1}} A(t_1)\ldots A(t_k)\,dt_k \quad (3.5)$$

We shall now derive expressions for the other coefficients in equation (0.2). The matrix function $A(t)$ is periodic, hence the matrix $X(t + \omega)$ is also the solution matrix of the system (0.1). Moreover, we have the identity $X(t + \omega) \equiv X(t)\,X(\omega)$, because the right-hand member of this identity is also the solution matrix of the system (0.1), and by virtue of the condition $X(0) = I_n$, both members are obviously equal when $t = 0$; hence they must be equal for any value of $t$ by the uniqueness theorem. Applying the above identity $n - 1$ times we find that $X(t + (n - 1)\,\omega) \equiv X(t)\,X^{n-1}(\omega)$ and substituting $t = \omega$ we obtain

$$X(n\omega) = X^n(\omega) \quad (3.6)$$

The coefficient $a_\nu (\nu = 1, \ldots, n - 1)$ in equation (0.2) equals the sum of the principal minors of $\nu$-th order of the matrix $X(\omega)$, that is

$$a_\nu \equiv \sigma_\nu(X(\omega)) \qquad (\nu = 1, \ldots, n-1;\ \sigma_1(X(\omega)) \equiv \mathrm{sp}\, X(\omega))$$

For an arbitrary matrix C the following relation:

$$\sigma_1(C^2) = \sigma_1^2(C) - 2\sigma_2(C)$$

is always valid, where $\sigma_1(C)$ is the sum of eigenvalues of the matrix C, and $\sigma_2(C)$ is the sum of products of these eigenvalues, taking two at a time.

Using (3.6) we obtain

$$a_2 \equiv \sigma_2(X(\omega)) = \frac{1}{2}\left[\sigma_1^2(X(\omega)) - \sigma_1(X(2\omega))\right]$$

or

$$a_2 = \frac{1}{2}\left[a_1^2 - a_1(2\omega)\right] \quad (3.7)$$

To derive $a_1(2\omega)$ we substitute in formulas for $a_1$, $2\omega$ for $\omega$.

In the general case the coefficients $a_\nu$ are obtained from the Waring formula (see, for example, [11], Section 127), which involves the trace of the matrix $X^k(\omega)$:

$$a_\nu = (-1)^\nu \sum (-1)^{j_1+\ldots+j_\nu} \frac{1}{1^{j_1}2^{j_2}\ldots\nu^{j_\nu} j_1!\ldots j_\nu!} a_1^{\,j_1} a_1(2\omega)^{j_2}\ldots a_1(\nu\omega)^{j_\nu}$$

The above sum is taken over all integral non-negative $j_1 \ldots j_\nu$, satisfying the condition

$$j_1 + 2j_2 + \ldots + \nu j_\nu = \nu,$$

$$a_1(k\omega) = \mathrm{sp}\, X(k\omega) = \mathrm{sp}\, X^k(\omega).$$

**4.** Let us investigate the following system of $m$ linear differential equations of the second order with sectionally continuous periodic coefficients:

$$\frac{d^2 y_i}{dt^2} + p_{i1}(t)\, y_1 + \ldots + p_{im}(t)\, y_m = 0, \quad p_{ij}(t) = p_{ji}(t) \qquad (i, j = 1, \ldots, m)$$

It can be written as one second-order vector differential equation:

$$\frac{d^2 \mathrm{y}}{dt^2} + \mathrm{P}(t)\,\mathrm{y} = 0, \qquad \mathrm{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \quad \mathrm{P}(t+\omega) = \mathrm{P}(t) = \| p_{ij}(t) \|_1^m \qquad (4.1)$$

The system (4.1) can be reduced to a special case of the system (1.1) if x is regarded as the direct sum of the vectors y and $d\mathrm{y}/dt$, and

$$\mathrm{H}(t) = \begin{pmatrix} \mathrm{P}(t) & 0 \\ 0 & \mathrm{I}_m \end{pmatrix}, \qquad \Lambda(t) = \mathrm{J}_{2m}\mathrm{H}(t) = \begin{pmatrix} 0 & \mathrm{I}_m \\ -\mathrm{P}(t) & 0 \end{pmatrix}$$

Let us write down equations (3.3):

$$\mathrm{X}_\varkappa(\omega) = \begin{pmatrix} \mathrm{B}_\varkappa & \mathrm{D}_\varkappa \\ \mathrm{F}_\varkappa & \mathrm{G}_\varkappa \end{pmatrix} \qquad (\varkappa = 1, 2, \ldots)$$

$$\mathrm{B}_1 = \mathrm{G}_1 = 0 \qquad \mathrm{D}_1 = \omega \mathrm{I}_m, \qquad \mathrm{F}_1 = -\int_0^\omega \mathrm{P}_1\, dt_1$$

$$\mathrm{B}_2 = -\int_0^\omega dt_1 \int_0^{t_1} \mathrm{P}_2\, dt_2, \qquad \mathrm{D}_2 = \mathrm{F}_2 = 0, \qquad \mathrm{G}_2 = -\int_0^\omega t_1 \mathrm{P}_1\, dt_1$$

$$\mathrm{B}_{2k-1} = \mathrm{G}_{2k-1} = 0$$

$$D_{2k-1} = (-1)^{k-1}\int_0^{\omega} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{2k-3}} t_{2k-2} P_2P_4 \ldots P_{2k-2}\, dt_{2k-2}$$

$$G_{2k-1} = (-1)^{k}\int_0^{\omega} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{2k-2}} P_1P_3 \ldots P_{2k-1}\, dt_{2k-1}$$

$$B_{2k} = (-1)^{k}\int_0^{\omega} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{2k-2}} P_2P_4 \ldots P_{2k}\, dt_{2k}, \quad D_{2k} = F_{2k} = 0$$

$$G_{2k} = (-1)^{k}\int_0^{\omega} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{2k-2}} t_{2k-1} P_1P_3 \ldots P_{2k-1}\, dt_{2k-1}$$

where $k = 2, 3, \ldots$ and for the sake of brevity $P_i = P(t_i)$.

The expression for $X(\omega)$ is:

$$X(\omega) = \begin{pmatrix} X_{11}(\omega) & X_{12}(\omega) \\ X_{21}(\omega) & X_{22}(\omega) \end{pmatrix}$$

$$X_{11}(\omega) = I_m - \int_0^{\omega} dt_1 \int_0^{t_1} P_2\, dt_2 + \int_0^{\omega} dt_1 \int_0^{t_1} P_2\, dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} P_4\, dt_4 - \ldots$$

$$X_{12}(\omega) = \omega I_m - \int_0^{\omega} dt_1 \int_0^{t_1} t_2 P_2\, dt_2 + \ldots$$

$$X_{21}(\omega) = -\int_0^{\omega} P_1\, dt_1 + \int_0^{\omega} P_1\, dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} P_3\, dt_3 - \ldots$$

$$X_{22}(\omega) = I_m - \int_0^{\omega} P_1\, dt_1 \int_0^{t_1} dt_2 + \int_0^{\omega} P_1\, dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} P_3\, dt_3 \int_0^{t_3} dt_4 - \ldots$$

The first coefficient in equation (1.2) equals

$$a_1 = \operatorname{sp} X(\omega) = 2m - a_1^{(1)} + a_1^{(2)} - \ldots + (-1)^k a_1^{(k)} + \ldots \qquad (4.2)$$

where

$$a_1^{(1)} = \operatorname{sp}\int_0^{\omega} dt_1 \int_0^{t_1} (P_1 + P_2)\, dt_2, \quad a_1^{(2)} = \operatorname{sp}\int_0^{\omega} dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} (P_1P_3 + P_2P_4)\, dt_4$$

$$a_1^{(k)} = \operatorname{sp}\Big[\int_0^{\omega} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{2k-1}} P_1P_3 \ldots P_{2k-1}\, dt_{2k} +$$

$$+\int_0^{\omega} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{2k-1}} \mathrm{P}_2\mathrm{P}_4 \ldots \mathrm{P}_{2k}\, dt_{2k}\Big] \tag{4.3}$$

It is well known that the trace of two square matrices does not depend on the order of the factors. It follows that the trace of the product of an arbitrary number of matrices remains invariant with respect to the cyclic permutation of the factors.

Let us perform the Liapunov transformation [ 10 ] on the formula (4.3). From the transformation

$$\int_0^{\omega} dt_1 \int_0^{t_1} \mathrm{P}_2\, dt_2 = \int_0^{\omega} \mathrm{P}_2\, dt_2 \int_{t_2}^{\omega} dt_1 = \int_0^{\omega} (\omega - t_2)\, \mathrm{P}_2\, dt_2 = \int_0^{\omega} (\omega - t_1)\, \mathrm{P}_1\, dt_1$$

we obtain

$$a_1^{(1)} = \mathrm{sp}\Big[\int_0^{\omega} \mathrm{P}_1\, dt_1 \int_0^{t_1} dt_2 + \int_0^{\omega} dt_1 \int_0^{t_1} \mathrm{P}_2\, dt_2\Big] =$$

$$= \mathrm{sp}\,\Big[\int_0^{\omega} t_1 \mathrm{P}_1\, dt_1 + \int_0^{\omega} (\omega - t_1)\, \mathrm{P}_1\, dt_1\Big] = \omega\, \mathrm{sp} \int_0^{\omega} \mathrm{P}(t)\, dt \tag{4.4}$$

A similar reduction could also be obtained in a general case. The multiple integrals of order $2k$ in formula (4.3) are reduced to one integral of $k$-th order. Indeed, both integrals in formula (4.3) could be considered as integrals over the variables $t_1,\ t_2,\ \ldots,\ t_{2k}$, satisfying the inequality $\omega > t_1 > t_2 > \ldots\ t_{2k} > 0$.

We shall perform the integration of the first integral with respect to the variables $t_2,\ t_4,\ \ldots,\ t_{2k}$ and of the second integral with respect to the variables $t_1,\ t_3,\ \ldots,\ t_{2k-1}$. Then, naming the $k$ remaining variables $t_1,\ t_2,\ \ldots,\ t_k$ we obtain

$$a_1^{(k)} = \mathrm{sp} \int_0^{\omega} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{k-1}} (\omega - t_1 + t_k)(t_1 - t_2) \ldots\ (t_{k-1} - t_k)\, \mathrm{P}_1\mathrm{P}_2 \ldots \mathrm{P}_k\, dt_k \tag{4.5}$$

The above formula is analogous to formula (10) of Liapunov [ 10 ]. Introducing the matrix function

$$\int \mathrm{P}(t)\, dt = \mathrm{Q}(t)$$

and performing integration of the first integral in formula (4.3) with respect to the variables $t_1,\ t_3,\ \ldots,\ t_{2k-1}$, and of the second integral with respect to the variables $t_2$, $t_4$, ..., $t_{2k}$, making the cyclic permutation in the product of matrices and naming the remaining variables $t_1$, $t_2$, ..., $t_k$ we obtain

$$a_1^{(k)} = \mathrm{sp}\int_0^{\omega} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{k-1}} (\mathrm{S} - \mathrm{Q}_1 + \mathrm{Q}_k)(\mathrm{Q}_1 - \mathrm{Q}_2)\ldots(\mathrm{Q}_{k-1} - \mathrm{Q}_k)\, dt_k \qquad (4.6)$$

where

$$\mathrm{S} = \int_0^{\omega} \mathrm{P}(t)\, dt \equiv \omega \mathrm{P}_{\mathrm{cp}}, \quad \mathrm{Q}_i = \mathrm{Q}(t_i)$$

The formula (4.6) is analogous to the formula (11) of Liapunov [ 10 ]. The first three terms in the series (4.2) could be expressed by formulas without multiple integrals. We write the periodic matrix function $P(t)$ in the form $P(t) = P_{cp} + \Phi(t)$, where the periodic matrix function $\Phi(t)$ satisfies the condition $\Phi_{cp} = 0$. Omitting derivations similar to those shown in Sections 4, 15, 16, 17 of [ 10 ], we establish the final result

$$a_1^{(1)} = \omega^2\, \mathrm{sp} \mathrm{P}_{\mathrm{cp}},\quad a_1^{(2)} = \mathrm{sp}\left[\frac{1}{12}\omega^4 \mathrm{P}_{\mathrm{cp}}^2 - \omega \int_0^{\omega} \dot{\Phi}^2\, dt\right]$$

$$a_1^{(3)} = \mathrm{sp}\left[\frac{1}{360}\omega^6 \mathrm{P}_{\mathrm{cp}}^3 - \frac{1}{6}\omega^3 \mathrm{P}_{\mathrm{cp}} \int_0^{\omega} \dot{\Phi}^2\, dt + \right. \qquad (4.7)$$

$$\left. + 4\omega \mathrm{P}_{\mathrm{cp}} \int_0^{\omega} \Phi^2\, dt - 2\omega \int_0^{\omega} \Phi\dot{\Phi}^2\, dt + \omega^2 \mathrm{P}_{\mathrm{cp}} \int_0^{\omega} \dot{\Phi}\Phi\, dt\right]$$

The last formula in (4.7) is analogous to the formula (37) of Liapunov [ 10 ]. In the latter the last term vanishes when $m = 1$ (the scalar case).

5. Let us assume that all elements of the symmetric matrix $P(t)$ are non-positive:

$$p_{ij}(t) \leqslant 0 \quad (i, j = 1, \ldots, m;\ 0 \leqslant t \leqslant \omega) \qquad (5.1)$$

All terms of the series (4.2) are positive; hence $a_1 > 2m$. By virtue of the inequality (1.4) when $\mu = 1$, we deduce the following theorem:

If all coefficients in the system (4.1) can assume only negative values (or zero), then the trivial solution of the system (4.1) is unstable.

The above theorem is an extension of Theorem I of Liapunov (Section

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* Translator's note: The author is using the subscript cp to denote the mean values of a function.

49 [ 2 ] ) for the system (4.1). Liapunov proved another theorem (Section 52 [ 2 ] ), which for the system (4.1) could be formulated as follows:

If all eigenvalues of the matrix $P(t)$ (which are real) are non-positive for all values of $t$ $(0 \leqslant t \leqslant \omega)$, then the trivial solution of the system (4.1) is unstable.

Liapunov mentioned that his Theorem I (Section 49 [ 2 ] ) is a special case of the above theorem. This raises the following question: Is it possible to obtain other criteria of instability for the system (4.1), which in the scalar case ($m = 1$) would coincide with Theorem I of Liapunov (Section 49 [ 2 ] )?

On the other hand the first Theorem in Section 5 is valid, whether the matrix $P(t)$ is symmetric or not. The essence of the problem is that in the most general case of the system (0.1), the conditions (2.1) are necessary conditions for the boundedness of all solutions. From the formula (3.5) it follows that if all elements of the matrix $A(t)$ are non-negative, then all terms of the series (3.4) are positive, and $a_1 > n$; hence the solution of the system (0.1) is unbounded. Moreover, in this case, by a Theorem of Perron ([ 12 ] , Chapter 13) there exist vector solutions $x_0(t)$ of equation (0.1) with non-negative coordinates such as $x_0(t + \omega) = \rho_0 x_0(t)$ $(\rho_0 > 1)$, where $\rho_0$ is the magnitude of the largest root of equation (0.2).

The statement that the trivial solution of the system (0.1) is unstable under the conditions

$$a_{ij}(t) \geqslant 0 \qquad (i, j = 1, \ldots, n;\ 0 \leqslant t \leqslant \omega) \tag{5.2}$$

is valid except when $\operatorname{sp} A(t) \equiv 0$, which includes the system (0.1) in canonical form. When $\operatorname{sp} A(t) \not\equiv 0$ then from the conditions (5.2) we obtain

$$\int_0^\omega \operatorname{sp} A(t)\, dt > 0$$

which determines the instability of the trivial solution (see introduction).

Certain results for the case $p_{ij}(t) \geqslant 0$ $(i, j = 1, \ldots, m;\ 0 \leqslant t \leqslant \omega)$, and for the general case, when the elements of the matrix $P(t)$ have alternating signs, are given in the author's doctoral dissertation "Certain problems of stability of periodic motions." (*Inst. Mekh. Akad. Nauk SSSR*, 1957).

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